مجلة العلوم البحتة والتطبيقية
Journal of Pure & Applied Sciences
www.Suj.sebhau.edu.ly ISSN 2521-9200
Received 27/03/2020 Revised 10/08/2020 Published online 05/10/2020

On Some Relations between the Hermite Polynomials and Some Well-Known Classical Polynomials and the Hypergeometric Function.

Haniyah Saed Ben Hamdin
Mathematics Department, Faculty of Science, Sirte University, Sirte, Libya.
Corresponding Author: h.saed1717@su.edu.ly

Abstract The connection between different classes of special functions is a very important aspect in establishing new properties of the related classical functions that is they can inherit the properties of each other. Here we show how the Hermite polynomials are related to some well-known classical polynomials such as the Legendre polynomials and the associated Laguerre polynomials. These relationships set up the connection between both kinds of classical orthogonal polynomials and grant us the ability to consider the theory of Hermite polynomials as a special case of the theory of Legendre and Laguerre polynomials. We show the confluent hypergeometric and the hypergeometric representations of Hermite polynomials. Thus the Hermite polynomials inherit the great advantage of carrying out their analytic continuation into any part of the complex z-plane. Furthermore, the hypergeometric representation enables us to develop the theory of the Hermite polynomials by implementing the general theory of the hypergeometric function. In this paper we have shown various types of formulae which link the Hermite polynomials to the Legendre polynomials. Some of these formulae are of integral form, operational form and an expansion form. Such diversity should grant us more flexibility to apply the Hermite polynomials in a variety of applications in mathematics, physics and engineering.

Keywords: Hermite Polynomials, Legendre Polynomials, Associated Laguerre Polynomials, Confluent Hypergeometric Function, Relations between Classical Polynomials.

حول بعض العلاقات بين متعددات حدود هيرمت وبعض متعددات الحدود الشهيرة والدالة فوق الهندسية

هنية سعد بن حمدين
قسم الرياضيات-كلية العلوم-جامعة سرت، ليبيا
للمراسلة: h.saed1717@su.edu.ly

الملخص متعددات الحدود المتعامدة بشكل عام ومتعددات حدود هيرمت ولجندر ولاقيرا على وجه الخصوص لها أهمية كبيرة في كافة المجالات الرياضية والفيزيائية والهندسية. هذه الورقة تهدف الى التعرف على كيفية ارتباط متعددات حدود هيرمت ببعض متعددات الحدود المتعامدة الشهيرة مثل متعددات حدود لجندر ومتعددات حدود لاقيرا المعممة. هذا الارتباط بين هذه الانواع من الدوال الخاصة المتعامدة يمنحنا القدرة على اعتبار نظرية كثيرات حدود هيرمت كحالة خاصة من نظرية متعددات حدود ليجندر ولاقيرا المعممة. تجدر الإشارة هنا إلى أن أحد العلاقات بين كثيرات حدود هيرمت ومتعددات حدود لجندر تتضمن الدالة فوق الهندسية. أيضا سنقدم التمثيل الفوق هندسي لمتعددات حدود هيرمت بدلالة الدالة فوق الهندسية وهذا يعطي كثيرات حدود هيرمت ميزة عظيمة وهي امكانية الحصول على الامتداد التحليلي لها في أي جزء من المستوى المركب z. علاوة على ذلك، فإن التمثيل الفوق الهندسي يتيح لنا أيضا تطوير نظرية كثيرات حدود هيرمت من خلال تطبيق النظرية العامة للدالة فوق الهندسية. في هذه الورقة، أظهرنا أنواعا مختلفة من الصيغ التي تربط كثيرات حدود هيرمت مع كثيرات حدود ليجندر وكثيرات حدود لاقيرا المعممة. الجدير بالذكر هو أن هذه الصيغ تأخذ قوالب رياضية متنوعة حيث أن بعضها يأخذ صورة تكاملية (سواء كان تكامل على المحور الحقيقي أو كنتوري في المستوى المركب) والاخرى تم تقديمها بدلالة المؤثرات والصيغ الاخرى تنشر أحد متعددات الحد ود بدلالة الاخرى. هذه الاثراء في التنوع في الاشكال الرياضية للصور المقدمة يتيح لنا مرونة أكثر في تطبيق كثيرات حدود هيرمت في العديد من التطبيقات الرياضية، الفيزيائية والهندسية.

الكلمات المفتاحية: متعددات حدود هيرمت، متعددات حدود لجندر، متعددات حدود لاقيرا المعممة، الدالة فوق الهندسية، العلاقات بين الدوال الخاصة.

Introduction

The classical polynomials in general and the Hermite polynomials [1, 2, 3, 4, 5] in particular are very important in many applications. The Hermite polynomials occur in wave mechanics in the treatment of harmonic oscillator, also appear in vibration theory and quantum theory of radiation. In this paper we shall introduce several relations that connect the Hermite polynomials with some well-known classical polynomials such as the Legendre polynomials [6, 7, 8, 9] and the associated Laguerre polynomials [6, 7, 8]. Furthermore we study the confluent hypergeometric and the hypergeometric representation [6, 7, 9] of the Hermite polynomials. The hypergeometric representations of the Hermite polynomials considerably benefit us in acquiring the analytic continuation of Hermite polynomials into any part of the complex z-plane [6]. In turn this should allow variety of applications for such polynomials.

Curzon [10] established many relations between the Hermite and the Legendre polynomials for general values of the index n. One of his relations is of real-integral type formula that will be shown in section 5. Furthermore, Curzon [10] introduced a relation that defines the Hermite polynomials as a contour integral of Legendre polynomials. However, such a relation was re−derived in a simpler fashion by Rainville [11] in terms of a real integral defining the Hermite polynomials in terms of the Legendre polynomials. Rainville [11] approach is based on taking the Laplace transform of a general formula of the generating function that was derived by Rainville [11] for most of the special functions.

On the other hand Araci et al [12] showed that the Legendre polynomials are proportional with the Hermite polynomials, their approach is based on the arithmetic properties of Legendre polynomials by making use of their orthogonally property. Furthermore a relationship between the Hermite polynomials and the Legendre polynomials has been established by Khammash et al. in [13] by implementing an integral transform representation. Such an approach has been proven beneficial in deriving most of the properties of the generalized Legendre polynomials [13]. An integral transform representation that connects the Hermite polynomials with the Legendre polynomials was obtained in [13] with the aid of operational methods [14, 15, 16] as it will be shown in section four.

This paper is structured as follows: in section one; we briefly set up some concepts that we need in the paper. These concepts consist of some needed identities on the convergent double series, the hypergeometric function, the Hermite polynomials, the Bessel functions, the associated laguerre polynomials, and finally the Legendre polynomials. Then the confluent hypergeometric representations of the Hermite polynomials and the associated Laguerre polynomials are respectively presented in section two and three. In section four, different formats of the relations between the Hermite and the Legendre polynomials will be introduced. Finally a conclusion is drawn in section five.

1 Preliminaries.

Here we shall introduce some necessary concepts, which we will need later on, such as a very important tool of dealing with double series.

1.1 Double Series Manipulations.

Here we shall introduce some elementary operations with the convergent double power series. Such operations will be needed later in rearrangement of series appearing later in the paper.

Theorem 1: For a non-negative integrals m, n and for a convergent power series φ, one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n-m). \qquad (1)$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n-2m), \qquad (2)$$

where [] is the greatest integer symbol defined as,

$$\left[\frac{n}{2}\right]=\begin{cases} n/2, & \text{for } n \text{ even,}\\ (n-1)/2, & \text{for } n \text{ odd.}\end{cases}$$

It should be noted that these identities can be taken in a reverse order, that is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n+m), \qquad (3)$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n+2m). \qquad (4)$$

Also note that a combination of the identities (1) and (2) yields,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n-m). \qquad (5)$$

1.2 The Hypergeometric Function.

In this section we shall introduce some functions that are used in this paper. Firstly, consider the series

$$1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)}\frac{z^n}{n!}, \qquad (6)$$

where z is a complex variable α or β and γ are parameters, which can take arbitrary real or complex values provided that $\gamma \neq 0,-1,-2,\dots$. If we let $\alpha=1$ and $\beta=\gamma$, then we get the elementary geometric series $\sum_{n=0}^{\infty}z^n$. The series (6) is called the Gauss hypergeometric series, which has great importance in mathematical analysis and its applications. It is very convenient to introduce the so called generalized factorial function [3, 5] or Pochhammer symbol $(\alpha)_n$ defined as,

$$(\alpha)_n=\prod_{k=1}^{n}(\alpha+k-1), (\alpha)_0=1, \quad \alpha\neq 0.$$

In terms of the Pochhammer symbol we can simplify the hypergeometric series (6) in the form,

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}.$$

Now we introduce the gamma function that is related to Pochhammer symbol.

Definition 1: For a non-negative number α, the gamma function $\Gamma(\alpha)$ is defined by the following Euler integral,

$$\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx, \qquad \alpha>0.$$

The Pochhammer symbol is related to the gamma function by the following relation.

Theorem 2: If a is neither zero nor a negative integer then,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a \neq 0, \pm 1, \pm 2, \dots, n = 0,1,2,\dots \qquad (7)$$

Next, we introduce some beneficial identities that we will need in our derivations in this article.

Example 1: show the identity:

$$\frac{(-n)_k}{n!} = \frac{(-1)^k}{(n-k)!}, \qquad 0 \leq k \leq n. \qquad (8)$$

Proof: since,

$$\frac{(-n)_k}{n!} = \frac{(-n)(-n+1)\dots(-n+k-1)}{n(n-1)\dots(n-k+1)(n-k)(n-k-1)\dots 3.2.1}, = \frac{(-1)^k n(n-1)\dots(n-k+1)}{n(n-1)\dots(n-k+1)(n-k)(n-k-1)\dots 3.2.1}.$$

Thus one obtains the required relation.

Example 2: show the identity:

$$2^{2n}\left(\frac{1}{2}\right)_n = \frac{(2n)!}{n!}. \qquad (9)$$

Proof: Using the relation (7) yields,

$$2^{2n}\left(\frac{1}{2}\right)_n = 2^{2n}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Calling the following relation of Gamma function,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi} . \qquad (10)$$

And the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, thus one obtains the desired result (9). And similarly, one has,

$$2^{2n+1}\left(\frac{3}{2}\right)_n = \frac{2(2n+1)!}{n!}. \qquad (11)$$

We shall denote the convergent hypergeometric series (6) by the notation $F(\alpha,\beta;\gamma;z)$ that is,

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n, \quad |z|<1, \gamma \neq 0,-1,-2,\dots (12)$$

Theorem 3: The confluent hypergeometric series which is defined by

$$\Phi(\alpha;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n n!}z^n, \qquad \forall\, z, \gamma \neq 0,-1,-2,\dots \; (13)$$

is convergent for all finite values of z . Thus the confluent hypergeometric function is an analytic for all finite values of z.

Example 3: Show that $(1-z)^a = F(a;z)$.

Solution: Since,

$$(1-z)^{-a} = 1 + az + \dots + a(a+1)(a+n-1)\frac{z^n}{n!} + \cdots$$

Thus,

$$(1-z)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!} = \Phi(a;-;z). \qquad (14)$$

1.3 Hermite Polynomials.

The Hermite polynomials are defined by the following generating function,

$$e^{(2xh-h^2)} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n, \qquad \forall \text{ finite } x,h. \qquad (15)$$

Since,

$$e^{(2xh-h^2)} = \left(\sum_{n=0}^{\infty}\frac{(2xh)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-h^2)^n}{n!}\right)$$

Now we may rewrite this double series using the identity (2) as,

$$e^{(2xh-h^2)} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-1)^k(2x)^{n-2k}}{k!\,(n-2k)!}h^n.$$

Equating the coefficients h^n of on both sides of this equation yields,

$$H_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^k n!\,(2x)^{n-2k}}{k!\,(n-2k)!}. \qquad (16)$$

Furthermore, the Hermite polynomials are defined by the following formula of Rodrigues type as,

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, n = 0,1,2,\dots \quad (17)$$

Theorem 4: For a non-negative integral n and a finite real x, one has

$$\frac{(2x)^n}{n!} = \sum_{k=0}^{[n/2]}\frac{H_{n-2k}(x)}{k!\,(n-2k)!}. \qquad (18)$$

Proof: From the generating function of the Hermite polynomials (15) we have

$$e^{2xh} = e^{h^2}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n,$$

$$\sum_{n=0}^{\infty}\frac{(2xh)^n}{n!} = \sum_{n=0}^{\infty}\frac{h^{2n}}{n!}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n.$$

Now we may rewrite this double series using the identity (2) as,

$$\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{H_{n-2k}(x)}{k!\,(n-2k)!}h^n.$$

Equating the coefficients h^n of on both sides, we reach the desired result (18).

1.4 Integral Representations of Hermite Polynomials [6].

Hermite polynomials have some beneficial representations expressed in terms of some familiar definite integrals as shown in the following lemma.

Lemma 1: The Hermite polynomials $H_{2n}(x)$ have the following integral representation for even index n,

$$H_{2n}(x) = \frac{2^{2n+1}(-1)^n e^{x^2}}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}u^{2n}\cos 2ux\; du, \quad (19)$$

and for odd index n,

$$H_{2n+1}(x) = \frac{2^{2n+2}(-1)^n e^{x^2}}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}u^{2n+2}\sin 2ux\; du. \qquad (20)$$

Proof: From calculus we borrow the following simple integral,

$$e^{-x^2} = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\cos 2ux\; du. \qquad (21)$$

Now differentiate the relation (21) $2n$ times with respect to the parameter x, one has

$$\frac{d^{2n}}{dx^{2n}}e^{-x^2}=\frac{2.2^n}{\sqrt{\pi}}\int_0^{\infty}e^{-u^2}\,u^{\mathrm{n}}\cos 2ux\;du, \quad (22)$$

Then we multiply both sides of (22) by the factor $(-1)^n e^{x^2}$ to recall the Rodrigues formula of Hermite polynomials (17), so one obtains the relation (19). In a similar way, we obtain the relation (20).

Result: Combining the both preceding integrals formulae (19) and (20), one has

$$H_n(x)=\frac{2^{2n}(-i)^n e^{x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2+2iux}\,u^{\mathrm{n}}\;du.$$

These integral formulae (19) and (20) will be useful in deriving relations between the Hermite and the associated Laguerre polynomials as shown later on.

1.5 Bessel Functions.

Here we shall present all the needed information about the Bessel functions which will be used in this paper. At first the Bessel differential equation takes the form,

$$x^2y''+xy'+(x^2-v^2)y=0.$$

Solving this equation about the regular singular point at $x=0$ using the Frobenius method [1, 2], one obtain the Bessel functions of the first kind of order v as,

$$J_v(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\;\Gamma(k+v+1)}\left(\frac{x}{2}\right)^{v+2k}. \quad (23)$$

Using this series, one could easily derive the following recurrence differential formula,

$$\frac{d}{dx}[x^vJ_v(x)]=x^vJ_{v-1}(x)\,, \quad (24)$$

which will be used later in this paper.

Example 5: Find $J_{1/2}(x)$ and $J_{-1/2}(x)$.

Since,

$$J_{1/2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\;\Gamma\left(k+\frac{1}{2}+1\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2k}.$$

$$=\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^{\frac{7}{2}}+\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^{\frac{9}{2}}-\cdots$$

$$J_{1/2}(x)=\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{7}{2}}+\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{9}{2}}-\cdots$$

$$J_{1/2}(x)=\frac{2}{\sqrt{\pi}}\sqrt{\frac{x}{2}}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right].$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\,sin\,x. \quad (25)$$

In a similar way we can obtain,

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x. \quad (26)$$

Example 6: Find $J_{3/2}(x)$ using $J_{1/2}(x),\;\; J_{-1/2}(x)$.

Using the following recurrence relation of Bessel functions,

$$J_{v+1}(x)=\frac{v}{x}J_v(x)-J'_v(x).$$

Thus one has,

$$J_{3/2}(x)=\frac{1}{2x}J_{1/2}(x)-J'_{1/2}(x).$$

From equations (25) and (26), we have

$$J_{3/2}(x)=\frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin x-\frac{d}{dx}\left(\sqrt{\frac{2}{\pi x}}\sin x\right),$$

$$J_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right). \quad (27)$$

Weber's Integral.

Theorem 5: The Weber's Integral is given as,

$$\int_0^{\infty}e^{-\alpha^2x^2}x^v\,J_v(\beta x)dx=\frac{\beta^v}{(2\alpha^2)^{v+1}}e^{-\frac{\beta^2}{4\alpha^2}},\alpha>0,\beta>0, \quad (28)$$

which can be proven by using the Bessel series (23) and then interchanging the order of the integral with the summation which is allowed due to the absolute convergence of the Bessel series (23) [9]. Finally recall the definitions of gamma function to reach the desired result.

1.6 Associated Laguerre Polynomials.

The associated Laguerre differential equation takes the form

$$xy''(x)+(\alpha+1-x)y'(x)+ny(x)=0.$$

Solving this equation about the regular singular point at $x=0$ using the Frobenius method [1, 2], one obtain the associated Laguerre polynomials $L_n^{\alpha}(x)$ as,

$$L_n^{\alpha}(x)=\sum_{k=0}^{n}\frac{\Gamma(\alpha+n+1)(-1)^k}{\Gamma(\alpha+\mathrm{k}+1)(n-k)!\,k!}x^k. \quad (29)$$

Also, the associated Laguerre polynomials $L_n^{\alpha}(x)$ are defined by the following formula of Rodrigues type as,

$$L_n^{\alpha}(x)=\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}),n=0,1,2,\ldots \quad (30)$$

1.7 Legendre Polynomials.

The Legendre polynomials $P_n(x)$ are defined by the following generating function as,

$$\sum_{n=0}^{\infty}P_n(x)\,h^n=(1-2xh+h^2)^{-\frac{1}{2}}.$$

Using this formula we will derive a very beneficial series expression of the Legendre polynomials that will be used later on. By rewriting the function $(1-2xh+h^2)^{-\frac{1}{2}}$ in terms of the hypergeometric function using the identity (14), one has

$$\sum_{n=0}^{\infty}P_n(x)\,h^n=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(2x+h)^n}{n!}h^{n+k}.$$

Now by using the binomial expansion of the term $(2x+h)^n$, we have

$$\sum_{n=0}^{\infty}P_n(x)\,h^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{1}{2}\right)_n(2x)^{n-k}}{k!\,(n-k)!}h^{n+k}.$$

Rearrange the double series using the identity (5), one has

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!} h^n.$$

Equating the coefficients of h^n of on both sides we have the following useful expression of $P_n(x)$ as,

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!}. \qquad (31)$$

2 The Confluent Hypergeometric Representations of Hermite Polynomials.

In this section, we shall obtain the confluent hypergeometric representations of Hermite polynomials with complex variable z. These representations have the advantage of carrying out the analytic continuation of Hermite polynomials into any part of the complex z-plane.

Replace n by the odd order $(2n+1)$ in the series expansion of Hermite polynomials (16), one has

$$H_{2n+1}(z) = \sum_{k=0}^{[(2n+1)/2]} \frac{(-1)^k (2n+1)!\,(2z)^{2n-2k+1}}{k!\;(2n-k+1)!}, n = 0,1,\ldots$$

where $[(2n+1)/2] = n$, Now replace k by $n-k$ to obtain the infinite series as

$$H_{2n+1}(z) = (2n+1)!\,(2z)\sum_{k=0}^{\infty} \frac{(-1)^{n-k}(2z)^{2k}}{(1+2k)!\;(n-k)!},$$

Call the identity (8), thus one has

$$H_{2n+1}(z) = (-1)^n \frac{(2n+1)!}{n!}(2z)\sum_{k=0}^{\infty} \frac{(-n)_k 2^{2k} z^{2k}}{(1+2k)(2k)!},$$

Now using the fact that $\frac{(2k)!}{2^{2k}} = \left(\frac{1}{2}\right)_k k!$, one has

$$H_{2n+1}(z) = (-1)^n \frac{(2n+1)!}{n!}(2z)\sum_{k=0}^{\infty} \frac{(-n)_k z^{2k}}{(1+2k)\left(\frac{1}{2}\right)_k k!},$$

Using the fact that $\left(\frac{3}{2}\right)_k = (1+2k)\left(\frac{1}{2}\right)_k$, one has

$$H_{2n+1}(z) = (-1)^n \frac{(2n+1)!}{n!}(2z)\sum_{k=0}^{\infty} \frac{(-n)_k z^{2k}}{\left(\frac{3}{2}\right)_k k!},$$

Using the notation of the confluent hypergeometric function (13), one has

$$H_{2n+1}(z) = (-1)^n \frac{(2n+1)!}{n!}(2z)\Phi\left(-n,\frac{3}{2};z^2\right).$$

Using the identity (11), one has,

$$H_{2n+1}(z) = (-1)^n 2^{2n+1}\left(\frac{3}{2}\right)_n z\,\Phi\left(-n,\frac{3}{2};z^2\right). \qquad (32)$$

In a similar fashion, we obtain the confluent hypergeometric representations of Hermite polynomials of even orders as,

$$H_{2n}(z) = (-1)^n \frac{(2n)!}{n!}\Phi\left(-n,\frac{1}{2};z^2\right).$$

Using the identity (9), one has,

$$H_{2n}(z) = (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n \Phi\left(-n,\frac{1}{2};z^2\right). \qquad (33)$$

These confluent hypergeometric representations of Hermite polynomials (32) and (33) are convergent everywhere in the complex plane. Another hypergeometric representation of Hermite polynomials can be obtained as,

$$H_n(z) = (2z)^n \left[1 - \frac{n(n-1)}{1!}(2z)^{-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2z)^{-4} + \cdots\right].$$

Since,

$$\frac{n(n-1)}{2^2} = \left(-\frac{n}{2}\right)\left(\frac{1-n}{2}\right) = \left(-\frac{n}{2}\right)_1\left(\frac{1-n}{2}\right)_1,$$

$$\frac{n(n-1)(n-2)(n-3)}{2^4} = \left(-\frac{n}{2}\right)\left(\frac{1-n}{2}\right)\left(\frac{2-n}{2}\right)\left(\frac{3-n}{2}\right), = \left(-\frac{n}{2}\right)\left(\frac{2-n}{2}\right)\left(\frac{1-n}{2}\right)\left(\frac{3-n}{2}\right), = \left(-\frac{n}{2}\right)_2\left(\frac{1-n}{2}\right)_2.$$

Thus one has,

$$H_n(z) = (2z)^n \sum_{k=0}^{[n/2]} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{1-n}{2}\right)_k}{k!}\left(-\frac{1}{z^2}\right)^k n = 0,1,2,\ldots$$

Now we can rewrite this series using the hypergeometric function as the following,

$$H_n(z) = (2z)^n \mathrm{F}\left(-\frac{n}{2},\frac{1-n}{2};-;-\frac{1}{z^2}\right). \qquad (34)$$

The hypergeometric representation of Hermite polynomials is convergent in the complex region $|z| > 1$.

3 The Confluent Hypergeometric Representation of the Associated Laguerre Polynomials.

Here we find the hypergemetric representation the associated Laguerre polynomials as well. Implementing the identity (8) in the series of the associated Laguerre polynomials (29), thus one has

$$L_n^\alpha(z) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)(-n)_k(-1)^k}{\Gamma(\alpha+\mathrm{k}+1)\mathrm{n}!\;k!} z^k.$$

Now using the property of gamma function (7) leads to,

$$\frac{\Gamma(\alpha+\mathrm{n}+1)}{\Gamma(\alpha+\mathrm{k}+1)} = \frac{\Gamma(\alpha+\mathrm{n}+1)/\Gamma(\alpha+1)}{\Gamma(\alpha+\mathrm{k}+1)/\Gamma(\alpha+1)} = \frac{(\alpha+1)_\mathrm{n}}{(\alpha+1)_\mathrm{k}}.$$

Thus

$$L_n^\alpha(z) = \frac{(\alpha+1)_\mathrm{n}}{n!}\sum_{k=0}^{n} \frac{(-n)_k z^k}{(\alpha+1)_\mathrm{k} k!}.$$

Using the notation of the confluent hypergeometric function (10), yields

$$L_n^\alpha(z) = \frac{(\alpha+1)_\mathrm{n}}{n!}\Phi(-n,\alpha+1;z). \qquad (35)$$

4 Results and Discussion.

4.1 Relations between the Hermite and the Associated Laguerre Polynomials.

Relations between the Hermite and the associated Laguerre polynomials can be derived using their confluent hypergeometric representations (33) and (35) respectively. Thus let $\alpha = \frac{1}{2}$ in the confluent hypergeometric representation of associated Laguerre polynomials (32), so

$$\Phi\left(-n,\frac{3}{2};x^2\right)=\left(\frac{3}{2}\right)_n\Big/ n!\; L_n^{1/2}(x^2). \qquad (36)$$

Now plug in equation (36) into the confluent hypergeometric of Hermite polynomials (32) to obtain,

$$H_{2n+1}(x)=(-1)^n 2^{2n+1}\left(\frac{3}{2}\right)_n x\frac{n!}{\left(\frac{3}{2}\right)_n} L_n^{1/2}(x^2).$$

Thus, we obtain this relation

$$H_{2n+1}(x)=(-1)^n 2^{2n+1} n!\, x\, L_n^{1/2}(x^2). \qquad (37)$$

In a similar analogy we obtain the following relation,

$$H_{2n}(x)=(-1)^n 2^{2n} n!\; L_n^{-1/2}(x^2). \qquad (38)$$

Thus the Hermite polynomials of even degree $H_{2n}(x)$ are just Laguerre polynomials $L_n^{-1/2}(x^2)$ up to a multiplicative constant.

Further relations between the Hermite and the associated Laguerre polynomials can be derived.

Theorem 6: The Hermite polynomials and the associated Laguerre polynomials are related by the following relations,

$$L_n^{3/2}(x^2)=\frac{(-1)^n}{n!\,x^2 2^{2(n+1)}}\left[\frac{H_{2n+1}(x)}{x}+\frac{H_{2n+2}(x)}{2}\right], \qquad (39)$$

and

$$L_n^{-3/2}(x^2)=\frac{(-1)^n}{n!\,2^{2n-1}}\left[xH_{2n-1}(x)+\frac{H_{2n-2}(x)}{2}\right]. \qquad (40)$$

Proof: To prove these relations, we start at the Weber integral (28) that involve the Bessel function of order ν by setting the following values,

$$\alpha=1,\qquad \beta=2\sqrt{x},\qquad x=\sqrt{w},\qquad \nu=n+\mu,$$

in the Weber integral (28), thus one has

$$e^{-x}x^{n+\mu}=\int_0^\infty e^{-w}\left(\sqrt{xw}\right)^{n+\mu} J_{n+\mu}\left(2\sqrt{xw}\right)dw. \qquad (41)$$

Now differentiate equation (41) with respect to x m times, one has

$$\frac{d^m}{dx^m}\left(e^{-x}x^{n+\mu}\right)=\int_0^\infty e^{-w}\frac{d^m}{dx^m}\left[\left(\sqrt{xw}\right)^{n+\mu}J_{n+\mu}\left(2\sqrt{xw}\right)\right]dw.$$

Now recall the recurrence relation (24), so

$$\frac{d^m}{dx^m}\left(e^{-x}x^{n+\mu}\right)=\int_0^\infty e^{-w}\,w^m\left(\sqrt{xw}\right)^{n-m+\mu}J_{n-m+\mu}\left(2\sqrt{xw}\right)dw. \qquad (42)$$

Let $n=m$ and multiply equation (42) by the factor $e^x x^{-\mu}/n!$, then recall the generating function of associated Laguerre polynomials (30), one has,

$$L_n^{\mu}(x)=\frac{e^x x^{-\mu/2}}{n!}\int_0^\infty e^{-w}\,w^{n+\frac{\mu}{2}}\left(\sqrt{xw}\right)^{n-m+\mu}J_{\mu}\left(2\sqrt{xw}\right)dw,\ \mu>1.$$

Now consider the special cases $\mu=\pm\frac{3}{2}$, hence

$$L_n^{3/2}(x)=\frac{e^x x^{-3/4}}{n!}\int_0^\infty e^{-w}\,w^{n+\frac{3}{4}}\,J_{3/2}\left(2\sqrt{xw}\right)dw.$$

Now recall $J_{3/2}(x)$ from equation (27), one has

$$=\frac{e^x x^{-3/4}}{n!}\int_0^\infty e^{-w}w^{n+\frac{3}{4}}\left[\sqrt{\frac{2}{2\pi xw}}\left(\frac{\sin\left(2\sqrt{xw}\right)}{\left(2\sqrt{xw}\right)}-\cos\left(2\sqrt{xw}\right)\right)\right]dw.$$

$$=\frac{e^x x^{-3/4}}{2x^{3/2}\sqrt{\pi}n!}\int_0^\infty\left[e^{-w}w^n\sin\left(2\sqrt{xw}\right)-w^{n+\frac{1}{2}}\cos\left(2\sqrt{xw}\right)\right]dw.$$

Making the substitutions $w=v^2, x\to x^2$, yields

$$L_n^{3/2}(x^2)=\frac{e^x}{x^3\sqrt{\pi}n!}\int_0^\infty e^{-v^2}v^{2n+1}\sin(2vx)\,dv-\frac{2e^x}{x^2\sqrt{\pi}n!}\int_0^\infty e^{-v^2}v^{2n+2}\cos(2vx)\,dv\,.$$

Finally recall the integral representations of Hermite polynomials (19) and (20); we end up at the relation (39). In a similar analogy we can obtain the relation (40) . The relations (41) and (42) are special results obtained by the author.

4.2 Relations between the Hermite and the Legendre polynomials.

Curzon [10] established many relations between the Hermite polynomials and the Legendre polynomials for unrestricted values of the index n. One of his relations is mentioned in the following theorem.

Theorem 7: For general values of the index n, we have the following real-integral type formula given as,

$$P_n(x)=\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^nH_n(xt)\ dt. \qquad (43)$$

Proof: This relation can be proven by recalling the series expansion of Hermite polynomials (16) as

$$\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^nH_n(xt)\ dt$$

$$=\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^n\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!\ (n-k)!}(2xt)^{n-2k}\ dt,$$

$$=\sum_{k=0}^{[n/2]}\frac{2^{n-2k+1}(-1)^k}{\sqrt{\pi}k!\ (n-k)!}x^{n-2k}\int_0^\infty e^{-t^2}t^{2\left(n-k+\frac{1}{2}\right)-1}\ dt,$$

where we were allowed to swap the order of the integral with the summation because of the absolute convergence of the Hermite polynomials. Finally, we call the Euler definition of Gamma function and the relation (13) to reach the desired formula (43). Next we show how to derive another formula in a different mathematical frame. Khammash et al. [13] derived an operational formula between the Hermite polynomials and the Legendre polynomials. Their approach starts from the formula (43) by making the change of variable, $w=xt$ thus one obtains,

$$P_n(x)=\frac{2}{n!\,x^{n+1}\sqrt{\pi}}\int_0^\infty e^{-\left(\frac{w}{x}\right)^2}w^{n+1}H_n(w)\ dw. \qquad (44)$$

Actually this relation can be seen as the Mellin integral transform of the following function,

$$F(y)=e^{-\left(\frac{y}{x}\right)^2}y^{n+1}H_n(y).$$

Then, they introduced the following operator which acts on the function $f(x)$ as,

$$e^{\mu x \frac{\partial}{\partial x}} f(x) = f(xe^{\mu}).$$

Now,

$$f(xt) = f(xe^{\ln t}) = t^{x\frac{\partial}{\partial x}} f(x),$$

$$H_n(xt) = H_n(xe^{\ln t}) = t^{x\frac{\partial}{\partial x}} H_n(x).$$

Thus equation (44) becomes,

$$P_n(x) = \frac{2\ H_n(x)}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} e^{n+x\frac{\partial}{\partial x}}\ dt.$$

Finally, they recall the gamma function definition to obtain the following operational formula,

$$P_n(x) = \frac{1}{n!\sqrt{\pi}} \Gamma\left[\left(n+1+x\frac{\partial}{\partial x}\right)/2\right]\ H_n(x). \qquad (45)$$

Next we show a relation between the Hermite polynomials and the Legendre polynomials in a different mathematical frame.

4.3 An expansion form of Legendre polynomials in terms of Hermite polynomials

Here we show how to expand the Legendre polynomials in a series of the orthogonal Hermite polynomials.

Theorem 8: The Legendre polynomials $P_n(x)$ can be expanded in a series of the Hermite polynomials $H_n(x)$ as,

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{F\left(-k, \frac{1}{2}+n-k; -; 1\right)(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x)}{k!\,(n-2k)!}. \qquad (46)$$

Proof: We start from the series of Legendre polynomials (31) as,

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\ (n-2k)!} h^n.$$

Rearrange this double series using (4) to obtain,

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} (2x)^n}{k!\ n!} h^{n+2k}.$$

Now recall the identity (18) to obtain,

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} H_{n-2j}(x)}{k!\ j!\,(n-2j)!} h^{n+2k}$$

Rearrange this double series using (4) to obtain,

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n,k,j=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k+2j} H_n(x)}{k!\ j!\,n!} h^{n+2k+2j}.$$

Rearrange this double series using (1) to obtain,

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n,k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{k-j} \left(\frac{1}{2}\right)_{n+k+j} H_n(x)}{k!\,(k-j)!\,n!} h^{n+2k}.$$

Now recall the following identities,

$$(-k)_j = \frac{(-1)^j k!}{(k-j)!}, \left(\frac{1}{2}\right)_{n+k+j} = \left(\frac{1}{2}+n+k\right)_j \left(\frac{1}{2}\right)_{n+k}.$$

Hence, one has

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n,k=0}^{\infty}\sum_{j=0}^{k} \frac{(-k)_j (-1)^k \left(\frac{1}{2}+n+k\right)_j \left(\frac{1}{2}\right)_{n+k} H_n(x)}{k!\,n!\,j!} h^{n+2k}.$$

Using the notation of the hypergeometric function (12), one has

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n,k=0}^{\infty} \frac{F(-k, \frac{1}{2}+n+k; -; 1)(-1)^k \left(\frac{1}{2}\right)_{n+k} H_n(x)}{k!\,n!} h^{n+2k}.$$

Finally using the identity (2) and the then equating the coefficient of h^n on both sides; one obtains the required relation (46).

It should be noted that the formula (46) expands the Legendre polynomials in terms of the Hermite polynomials with a coefficient that involves the hypergeometric function of a constant argument $(x = 1)$.

5. Conclusion

To conclude we have presented some relations that link the Hermite polynomials to some well-known classical polynomials such as the Legendre polynomials and the associated Lagueree polynomials. Furthermore, we have introduced the confluent hypergeometric (32) and (33) and the hypergeometric representation (34) of the Hermite polynomials. Since the confluent hypergeometric series is convergent everywhere in the complex z plane, thus the Hermite polynomials inherit this great feature which reward us more adaptability in dealing with Hermite polynomials and allowing more applications of such polynomials. To sum up, in this paper we have shown various types of formulae which connect the Hermite polynomials with some well-known classical polynomials. It should be noted that these formulae hold distinct mathematical frames. Some of these relations are of integral (either real or contour) form (43), an operational form (45) and series expansion form (46) with a coefficient that involves the hypergeometric function. One could claim that by rewriting the Hermite polynomials in terms of the Legendre polynomials, one could treat the former as a special case of the latter polynomials. As future work, this approach can be extended to other well-known classical polynomials, such as Bessel, Chebychev, Jacobi, Gegenbauer polynomials, etc.

References

[1]- Arfken, G., **Mathematical Methods for Physicists**, 3rd ed., Academic Press, Inc. 1985.

[2]- Rainville, E.D., **Intermediate Differential Equations**, 2nd ed. New York : The Macmillan Co., 1964.
[3]- Abramowitz M and Stegun A. **Handbook of Mathematical Functions**, 9th ed., New York: Dover Publications Inc., 1968.
[4]- Andrews GE, Askey R, Roy R. **Special Functions**. Cambridge Univ. Press, 1999.
[5]- 6. Rainville ED. **Special Functions**. The Macmillan Co., New York, 1960.
[6]- Lebedev NN. **Special Functions and their Applications.** Dover Publication, Inc., New York, 1972.
[7]- Sneddon, I.N. **Special Functions of Mathematical Physics and Chemistry**, 3rd ed. New York: Longman,1980.
[8]- Rainville, E.D., Phillip E.B, **Elementary Differential Equations**, 6th ed. New York : Macmillan Publishing Co., Inc. 1981.
[9]- Bell, W.W., **Special Functions for Scientists and Engineers**, New York,1968.
[10]- Curzon, H. E. J., **On a Connexion between the Functions of Hermite and the Functions of Legendre**, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 236-259.
[11]- Rainville ED. **Notes on Legendre polynomials**, Bull. Amer. Math. Soc., Volume 51, Number 4, 1945, 268-271.
[12]- Araci S., Acikgoz M., Bagdasaryan A., and Şen E.. **The Legendre Polynomials Associated with Bernoulli, Euler, Hermite and Bernstein Polynomials**. Turkish Journal of Analysis and Number Theory 1, no. 1 (2013): 1-3.
[13]- Ghazi S. Khammash and A. Shehata. **A study on Some Properties of Legendre Polynomials and Integral Transforms**. Int. J. Open Problems Compt. Math., Vol. 5, no. 2 (2012).
[14]- Dattoli G., Torre A. and Carpanese M., **Operational rules and arbitrary order Hermite generating functions**, J. Math. Analysis Appl., 227 (1998), 98-111.
[15]- Dattoli G., Torre A. Lorenzutta S., **Operational identities and properties of ordinary and generalized special functions**, J. Math. Analysis Appl. 236 (1999), 399-414.
[16]- Ditkin, V.A., Prudnikov A., **Integral Transforms and Operational Calculus**, Pergamon-Press, Oxford, 1965.