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The HK Transform: A Laplace-Based Method for Population Growth and Decay Models

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ABSTRACT

This study examines the use of the HK transform in classical problems in population dynamics, particularly growth and decay processes. Theoretical sociology, applied chemistry, quantitative biology, physical sciences, and financial modelling are among the many disciplines in which these problems are highly relevant. The main aim of this study is to present a range of real-world case studies and to evaluate the effectiveness and accuracy of the HK transform as a reliable analytical tool for solving these differential equations. These examples are selected to demonstrate the transform's theoretical significance as well as its practical applications across several fields. The study offers a reliable and often simpler alternative to traditional techniques by using the HK transform. The results suggest improved efficiency and reliability in modelling these problems accurately. Overall, this study indicates that the HK transform is a useful tool for researchers studying rates of change in dynamic systems.

تحويل HK: طريقة مبنية على تحويل لابلاس لحل مسائل النمو والاضمحلال السكاني

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الكلمات المفتاحية:
تحويل HK .
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مشكلة النمو.
مشكلة الاضمحلال.

الملخص

تدرس هذه الدراسة الاستخدام الفريد لتحويل (HK transform) HK للمشكلات الشائعة في ديناميكيات السكان، وبالأخص ظواهر النمو والتضاؤل. تُعد هذه المشكلات ذات أهمية قصوى في مجموعة واسعة من التخصصات، بما في ذلك علم الاجتماع النظري، والكيمياء التطبيقية، وعلم الأحياء الكمي، والعلوم الفيزيائية، والنمذجة المالية. تتمثل أهدافنا الرئيسية في نقطتين: أولاً، إجراء تقييم دقيق لقدرة ودقة تحويل HK كأداة تحليلية موثوقة لمعالجة هذه الحالات التفاضلية؛ وثانياً، تقديم مجموعة واسعة من دراسات الحالة الواقعية. لقد تم اختيار هذه الأمثلة بعناية لإظهار القيمة النظرية العالية للتحويل، فضلاً عن فائدته العملية المباشرة في مجالات متعددة. تقدم هذه الدراسة، من خلال توظيف تحويل HK، بديلاً قوياً وغالباً ما يكون أبسط للتقنيات التقليدية. تُظهر النتائج بوضوح الفعالية والموثوقية الفائقة لتحويل HK في النمذجة الصحيحة. وفي الختام، تقترح هذه الدراسة أن تحويل HK هو أداة بالغة الأهمية للباحثين الذين يدرسون معدلات التغير في الأنظمة الديناميكية.

1. Introduction

Differential equations are commonly used in many fields of engineering and science to describe complex physical phenomena, and mathematical models are essential for improving the accuracy of biotechnological processes [1], [2]. Integral transformations are currently recognised as one of the most important mathematical techniques for solving complex problems across science, technology, engineering, and finance, primarily due to their ability to simplify these problems [3], [4]. One important advantage of integral transformations is their ability to provide precise solutions without the need for time-consuming computations. Academics are increasingly focused not only on developing novel integral transforms but also on their practical applications across a wide range of fields and equations. Recently, this application-focused research has been initiated by many scientists [5], [6].

Researchers in [7], [8] employed the Sawi transformation and numerical approximation methods to solve higher-order fractional differential equations both exactly and approximately. To address problems of population growth and decay, a new transform, known as the Rishi transform, was introduced in [9], [10], and has also been applied to solve higher-order fractional differential equations.

Applying a practical approach to evaluating a country's future

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population is now extremely important. A population model is a mathematical representation used in population dynamics. A robust population model can be developed using several methods. Among these, the logistic differential equation model developed by Verhulst (1838) and the Malthusian exponential model proposed by Thomas Robert Malthus (1766–1834) are well known [11].

The exponential model provides unreliable and inaccurate predictions of future population growth since it does not account for environmental limitations. As a result, it may suggest unrealistically large population growth over time. The population grows exponentially, doubling approximately every 25 years or more, which is a major limitation of the Malthusian model [12].

The application of innovative integral transforms to model population dynamics has led to important academic developments. In particular, Patil et al. used various techniques to address growth and decay problems, including the Kushare transform in 2022 [13], [14]. Additionally, the Emad Sara transform has been successfully applied to similar population growth and decay problems, indicating a broader trend in using different mathematical frameworks to solve these fundamental biological models [15]. Some studies have also been used to solve growth and decay models and have suggested various results [16]– [18].

Researchers can make well-informed decisions to balance ecological sustainability and economic demands by applying the logistic equation under different conditions. This ensures that resources are used efficiently without depleting natural systems [19], [20].

Population growth has been extensively studied, with early work by Thomas Robert Malthus (1798) [21] highlighting potential resource shortages due to exponential growth. The Malthus model is applied in this context, which is [1].

$$\frac{du(t)}{dt} = ru(t)$$

This law states that the growth rate is intrinsic growth rate and that it is proportionate to the current population (r). Our solution is $u(t) = u(0)e^{rt}$, which we may acquire by applying the variable separable approach. We can infer from this equation that the population graph is growing exponentially. In its early stages, this model makes sense. Verhulst's logistic model (1838) and subsequent advancements by Lotka (1925) and others have provided frameworks for understanding how populations stabilize when resources are limited. Recent research continues to explore demographic factors and their impact on resource management. Thus, the logistic model takes the place of the Malthus model[1], [12], [22].

$$\frac{dp(t)}{dt} = rp(t)\left(1 - \frac{p(t)}{K}\right)$$

where K is carrying capacity, meaning that when the population increases and gets closer to K, the growth will eventually reach zero and there will be a limit. Protozoa and bacterial species frequently engage in predator-prey interactions. The predator-prey concept is therefore crucial.

Research on the application of integral transforms in complex population dynamics modeling and decay problems is still lacking despite these advancements. Prior research has mostly focused on solving specific physical problems or general differential equations, with little consideration given to interdisciplinary settings that include growth and decay processes. This gap highlights the need for additional research to develop and assess integral transforms, like the HK Transform, to effectively manage intricate population dynamics. In our work, growth and decay problems are addressed using the HK Transform.

In[23], the authors have accurately demonstrated the difference between the logistic and exponential growth models using data, focusing on the population of Iraq based on the latest census and projections.

As the examples mentioned above show, a number of integral transformations have been created and used to handle different mathematical models, especially growth and decay models. To solve these models more successfully and precisely, there is still a great deal of space for study and use of other transformations.[24]–[28]

In this study, we use the HK transform to find the analytical solutions of differential equations for growth and decay we have suggested some result about age of fossil and radioactive decay which is rarely described in the previously research.

Fossil age is of great importance in archeology and history, and radioactivity is widely used in the physical and applied sciences.

In this paper, we have studied them and shown the results in detail.

The structure of the paper is as follows: Section 2 contains preliminary information about HK transform and its properties. In section three, growth and decay problems are addressed using the HK transform. Applications are shown both analytically and graphically in section four, with the conclusion given in the last section.

2. Preliminaries: $\mathcal{HK}$ transform Properties

We have described the transform and its properties in this section. [29]. Using a number of further investigations, we will find a summary of the new transform's fractional formula:

Definition 2.1 [$\mathcal{HK}$ transforms,[29]]. The HK transform of $s(t)$ denoted by the operator $HK(.)$, $s(t)$ and $s_1, s_2 > 0$,defined in the interval, $(0,\infty)$ is given by $HK\{s(t)\} = \frac{1}{\sqrt{z}}\int_0^\infty e^{-tz}s(t)dt = S(z), t > 0 , s_1 < z < s_2$.

Where $s(t)$ is an exponential order piecewise continuous function.

Definition 2.2 [30] If a function $f(t)$ is defined on any interval $[a, b]$ and The interval can be divided into a limited number of smaller intervals that $f(t)$ is continuous, In that case, the function is considered piecewise continuous. To put it another way, $f(t)$ can only have a finite number of finite discontinuities if it is piecewise continuous.

Definition 2.3[31] A function $f(t)$ is said to be of exponential order if $\lim_{t\to\infty} e^{-st}f(t) = 0$.

Property 1[29] HK transform for some basic functions:

1. Consider the function $s(t) = t$, from the definition of $\mathcal{HK}$ Transform we have

$\mathcal{HK}\{s(t)\} = \frac{1}{\sqrt{z}}\int_0^\infty e^{-tz}s(t)dt = \frac{1}{\sqrt{z}}\int_0^\infty te^{-tz}dt$, Let $u = t \Rightarrow du = dt$ and let $dv = e^{-tz}dt \Rightarrow v = \frac{-1}{z}e^{-tz}$, then $\Rightarrow \mathcal{HK}\{t\} = \frac{1}{\sqrt{z}}\left\{\frac{-t}{z}e^{-tz}\Big|_0^\infty + \int_0^\infty \frac{1}{z}e^{-tz}dt\right\} = \frac{1}{\sqrt{z}}\left\{-\frac{1}{z^2}e^{-tz}\right\}_0^\infty = \frac{1}{z^2\sqrt{z}} = \frac{1}{z^{\frac{5}{2}}}$.

2. Consider the function $s(t) = e^{at}$, from the definition of $\mathcal{HK}$-Transform we have

$\mathcal{HK}\{s(t)\} = \frac{1}{\sqrt{z}}\int_0^\infty e^{-tz}s(t)dt = \frac{1}{\sqrt{z}}\int_0^\infty e^{-tz}e^{at}dt = \frac{1}{\sqrt{z}}\int_0^\infty e^{-t(z-a)}dt = \frac{1}{\sqrt{z}}\cdot\frac{-1}{z-a}e^{-t(z-a)}\Big]_0^\infty = \frac{1}{\sqrt{z}(z-a)} = \frac{1}{z^{\frac{3}{2}}-az^{\frac{1}{2}}}$.

Property 2 [29] The HK transformation of few basic functions:

$s(t), t > 0$	$\mathcal{HK}\{s(t)\} = S(z)$	$s(t), t > 0$	$\mathcal{HK}\{s(t)\} = S(z)$
c	$cz^{-\frac{3}{2}}$	$sinat$	$\frac{a}{\sqrt{z}(z^2+a^2)}$
e^{at}	$\frac{1}{\sqrt{z}(z-a)}$	$cosat$	$\frac{\sqrt{z}}{z^2+a^2}$
$t^n, n \in N$	$\frac{n!}{z^{n+\frac{3}{2}}}$	$sinhat$	$\frac{a}{\sqrt{z}(z^2-a^2)}$
$t^\beta, \beta > -1, \beta \in R$	$\frac{\Gamma(\beta+1)}{z^{\beta+3/2}}$	$coshat$	$\frac{\sqrt{z}}{z^2-a^2}$

Property 3 [29] The HK transformation inverse:

$S(z)$	$\mathcal{HK}^{-1}\{S(z)\} = s(t)$	$S(z)$	$\mathcal{HK}^{-1}\{S(z)\} = s(t)$
$\frac{c}{z^{\frac{3}{2}}}$	c	$\frac{a}{\sqrt{z}(z^2+a^2)}$	$sinat$
$\frac{1}{\sqrt{z}(z-a)}$	e^{kt}	$\frac{\sqrt{z}}{z^2+a^2}$	$cosat$
$\frac{n!}{z^{n+\frac{3}{2}}}$	$t^n, n \in N$	$\frac{a}{\sqrt{z}(z^2-a^2)}$	$sinhat$
$\frac{\Gamma(\beta+1)}{z^{\beta+3/2}}$	$t^\beta, \beta > -1, \beta \in R$	$\frac{\sqrt{z}}{z^2-a^2}$	$coshat$

Property 4 :[29] Both the $\mathcal{HK}$ transform and its inverse are linear operators.

Property 5 [29] The integer order derivative $\mathcal{HK}$ transformation formula of $s(t)$ is:

1. $\mathcal{HK}(s'(t)) = zS(z) - z^{-\frac{1}{2}}s(0)$,
2. $\mathcal{HK}\{s''(t)\} = z^2 s(z) - z^{\frac{1}{2}}s(0) - z^{-\frac{1}{2}}s'(0)$,
3. $\mathcal{HK}\{s^{(n)}(t)\} = z^n S(z) - \sum_{k=0}^{n-1} z^{k-\frac{1}{2}} s^{(n-1-k)}(0)$.

3. Population Growth Model

The population growth of a plant, cell, organ, or species can be quantitatively defined using a first order ordinary linear differential equation [1], [32] as $\frac{df}{dt} = mf$ (2)

And the initial condition $f(t_0) = f_0$ such that $m \in R^+$, The population at time t is denoted by f, while the initial population at time $t = t_0$ is denoted by f_0. Equation represents the Malthusian law of population growth (2). The following first-order ordinary linear differential equation provides a mathematical definition for the substance's decay problem. [1], [32] as $\frac{df}{dt} = -mf$ (3) with initial condition $f(t_0) = f_0$ where f is the substance's quantity at that moment. f_0 is the substance's starting quantity at time $t = t_0$, and t, m is a positive real integer.

Since the mass of the material decreases with time, the derivative in equation (3) has a negative sign in the right-hand side $\frac{df}{dt}$ must be negative.

3.1 $\mathcal{HK}$ transform for population growth problem

The population grow problem is addressed with the HK transform in this section. Equation (2) presents its mathematical formulation. Taking $\mathcal{HK}$ transform for both sides of (2), we have

$$\mathcal{HK}\left\{\frac{df}{dt}\right\} = m\mathcal{HK}\{f(t)\}$$

Using the property 4, $\mathcal{HK}$ transform of derivative of function, on (2), we have $\mathcal{HK}\{f^n(t)\} = z^n F(z)$

$\mathcal{HK}\{f^n(t)\} = z^n F(z) - \sum_{m=0}^{n-1} z^{m-\frac{1}{2}} f^{n-1-m}(0)$, we have $n = 1$. So, we obtain

$$zF(z) - \sum_{m=0}^{1-1} z^{m-\frac{1}{2}} f^{(1-1-m)}(0) = m\mathcal{HK}\{f(t)\}$$

$$zF(z) - \frac{1}{\sqrt{z}} f(0) = mF(z) \quad (4)$$

Using initial condition $f(0) = f_0$ in (4) and on simplification, we have

$$zF(z) - mF(z) = \frac{1}{\sqrt{z}} f(0)$$

$$(z-m)F(z) = \frac{1}{\sqrt{z}} f_0, \Rightarrow \Rightarrow F(z) = \frac{f_0}{\sqrt{z}(z-m)} \quad (5)$$

Operating inverse $\mathcal{HK}$ transform on both sides of (5) and using property 2 , we obtain $\mathcal{HK}^{-1}\{F(z)\} = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z-m)}\right\} \Rightarrow f(t) = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z-m)}\right\}$

$$f(t) = f_0\, \mathcal{HK}^{-1}\left\{\frac{1}{\sqrt{z}(z-m)}\right\} \Rightarrow f(t) = f_0 e^{mt} \quad (6)$$

This is the population's necessary amount at time t.

3.2 $\mathcal{HK}$ transform for decay problem

This part introduces the HK transform for the decay issue, which may be stated mathematically using (3). HK transform applied to both sides of (3) yields

$$\mathcal{HK}\left\{\frac{df}{dt}\right\} = -m\mathcal{HK}\{f(t)\} \quad (7)$$

then using the property 1, $\mathcal{HK}$ transforms of derivative of function, on (7), we have

$\mathcal{HK}\{f^n(t)\} = z^n F(z) - \sum_{m=0}^{n-1} z^{m-\frac{1}{2}} f^{n-1-m}(0)$, we have $n = 1$

So, we obtain $zF(z) - \sum_{m=0}^{1-1} z^{m-\frac{1}{2}} f^{(1-1-m)}(0) = -m\mathcal{HK}\{f(t)\}$

$$zF(z) - \frac{1}{\sqrt{z}} f(0) = -mF(z) \quad (8)$$

From initial condition $f(t_0) = f_0$ in (8) and by simplify, we get

$$(z+m)F(z) = \frac{1}{\sqrt{z}} f_0, \Rightarrow F(z) = \frac{f_0}{\sqrt{z}(z+m)} \quad (9)$$

applying inverse $\mathcal{HK}$ transform for both sides of (9), we obtain

$$\mathcal{HK}^{-1}\{F(z)\} = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z+m)}\right\} \Rightarrow f(t) = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z+m)}\right\}$$

$$f(t) = f_0\mathcal{HK}^{-1}\left\{\frac{1}{\sqrt{z}(z+m)}\right\} \Rightarrow f(t) = f_0 e^{mt} \quad (10)$$

This is the population's necessary size at time t.

4. Applications:

A selection of examples presented in this section demonstrate the applicability of the HK transform for population growth and decay problems.

Application 1:
The pace at which a city's population grows is proportionate to the total number of people who now call it home. If the population doubles after five years and reaches 40,000 after seven, calculate how many people lived in the city at first.

Solution: This issue may be expressed mathematically as follows:

$$\frac{df}{dt} = mf \quad (11).$$

where m is the proportionality constant and f is the number of people living in cities at any given moment t. Assume that f_0 is the starting population of the city at $t = 0$.

Using property 1, HK transforms of the function's derivative on (11), we obtain

$\mathcal{HK}\{f^n(t)\} = z^n F(z) - \sum_{m=0}^{n-1} z^{m-\frac{1}{2}} f^{n-1-m}(0)$, we have $n = 1$

So, we obtain $zF(z) - \sum_{m=0}^{1-1} z^{m-\frac{1}{2}} f^{(1-1-m)}(0) = m\mathcal{HK}\{f(t)\}$

$$zF(z) - \frac{1}{\sqrt{z}} f(0) = mF(z)$$

From initial condition $f(t_0) = f_0$ in above equation and by simplify, we get $(z-m)F(z) = \frac{1}{\sqrt{z}} f_0, \Rightarrow \Rightarrow F(z) = \frac{f_0}{\sqrt{z}(z-m)}$

Applying inverse $\mathcal{HK}$ transform for above equation, we get

$$\mathcal{HK}^{-1}\{F(z)\} = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z-m)}\right\} \Rightarrow f(t) = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z-m)}\right\}$$

$$f(t) = f_0\mathcal{HK}^{-1}\left\{\frac{1}{\sqrt{z}(z-m)}\right\} \Rightarrow f(t) = f_0 e^{mt} \quad (12)$$

Now at $t = 4, f = 2f_0$, so putting this in (12), we get $2f_0 = f_0 e^{5m} \Rightarrow e^{5m} = 2 \Rightarrow m = 0.2ln2 = 0.1386$ (13)

Now using the condition at $t = 7, f = 40{,}000$, in (12), we obtain

$$40{,}000 = f_0 e^{7m} \quad (14)$$

Put the value of m from (13) in (14), we have

$40{,}000 = f_0 e^{7\times0.1386} \Rightarrow 40{,}000 = 8.9981 f_0 \Rightarrow f_0 \cong 4445$

This is the minimum number of individuals who must live in the city at first.

Figure 1 shows the exact solution of application 1.

Application 2: It is commonly recognized that the concentration of a radioactive element affects how quickly it decays. Calculate the half-life of the radioactive material if there were initially 100 milligrams of it and it was observed that the material had lost 25% of its initial mass after five hours.

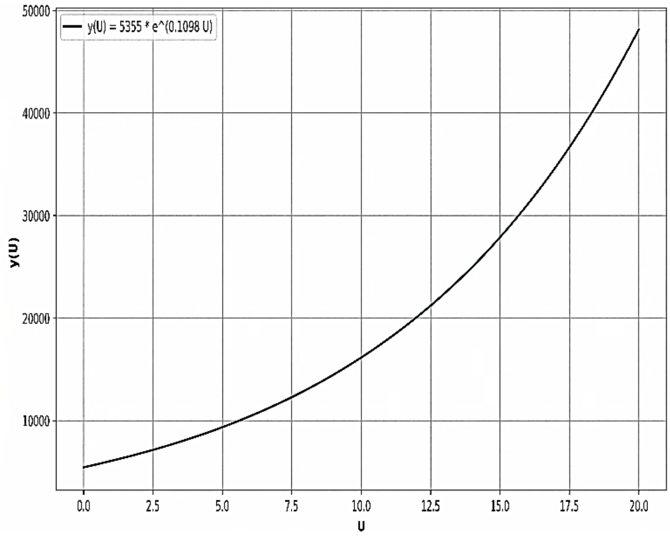

Figure 1. Exact Solution for Problem 1

Solutions: The following is a mathematical expression for this problem: $\frac{df(t)}{dt} = -mf(t)$ (15)

where f is the amount of radioactive material at time t and m is the proportionality constant. Let f_0 be the initial concentration of the radioactive material at $t = 0$.

By using the property 1, $\mathcal{HK}$ transform of derivative of function, on (15), we get

$\mathcal{HK}\{f^n(t)\} = z^n F(z) - \sum_{m=0}^{n-1} z^{m-\frac{1}{2}} f^{n-1-m}(0)$, we have $n = 1$

So, we obtain $zF(z) - \sum_{m=0}^{1-1} z^{m-\frac{1}{2}} f^{(1-1-m)}(0) = -m\mathcal{HK}\{f(t)\}$

$$zF(z) - \frac{1}{\sqrt{z}} f(0) = -mF(z)$$

From Using initial conditions $f(t_0) = f_0$ in previously equation and on simplify, we have

$(z+m)F(z) = \frac{1}{\sqrt{z}} f_0 \,, \Rightarrow \; \Rightarrow F(z) = \frac{f_0}{\sqrt{z}(z+m)}$

Operating inverse $\mathcal{HK}$ transform on both sides of above equation, we get $\mathcal{HK}^{-1}\{F(z)\} = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z+m)}\right\} \Rightarrow f(t) = \mathcal{HK}^{-1}\left\{\frac{f_0}{\sqrt{z}(z+m)}\right\}$

$$f(t) = f_0 \mathcal{HK}^{-1}\left\{\frac{1}{\sqrt{z}(z+m)}\right\} \Rightarrow f(t) = f_0 e^{mt} \qquad (16)$$

Now at $t = 5$, the radioactive substance has lost 25 percent of its original mass 100 mg so $f = 100 - 25 = 75$, using this in (16), we have $75 = 100e^{-5m} \Rightarrow e^{-5m} = 0.75 \Rightarrow m = -0.2\, ln\, 0.75 = 0.057$ (17)

We needed t when $f = \frac{f_0}{2} = \frac{100}{2} = 50$ so from (18), we get

$50 = 100e^{-mt}$ (18)

using the value of m from (17) in (18), we get

$50 = 100e^{-0.057t} \Rightarrow e^{-0.057t} = 0.5 \Rightarrow t = -\frac{1}{0.057} ln 0.5 \Rightarrow t = 12.16$ hours.

This is the radioactive substance's necessary half-time.

Figure 2 shows the exact solution of application 2.

Application 3: A fossilized bone is found to contain 0.1% of its original amount of $Carbon - 14$. Determine the age of the fossil. With initial condition $y(5730) = \frac{1}{2} y_0$.

Solution: The starting point is again after solve the D.E $\frac{dy}{dt} - ky = 0$

We have $\mathcal{HK}$ transforms of derivative of function, so we get

$\mathcal{HK}\{y^n(t)\} = z^n Y(z) - \sum_{m=0}^{n-1} z^{m-\frac{1}{2}} y^{n-1-m}(0)$, we have $n = 1$

So, we obtain $zY(z) - \sum_{m=0}^{1-1} z^{m-\frac{1}{2}} y^{(1-1-m)}(0) = k\mathcal{HK}\{y(t)\}$

$zY(z) - \frac{1}{\sqrt{z}} y(0) = kY(z)$

From initial condition $y(0) = y_0$ in above equation and by simplify, we get $(z-k)Y(z) = \frac{1}{\sqrt{z}} y_0 \,, \Rightarrow \; \Rightarrow Y(z) = \frac{y_0}{\sqrt{z}(z-k)}$

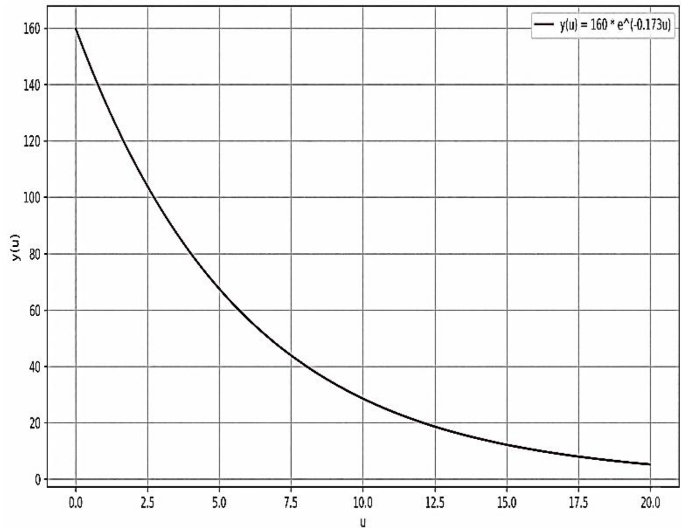

Figure 2. Exact Solution for Problem 2

When we apply the inverse HK transform to the equation above, the result is $y(t) = y_0 e^{kt}$. We use this information to calculate the decay constant k's value.

$\frac{1}{2} y_0 = y(5730)$ or $\frac{1}{2} y_0 = y_0 e^{5730k}$. The last equation implies $5730k = ln\frac{1}{2} = -ln2$ and so we get $k = -(ln2)/5730 = -0.00012097$. Therefore $y(t) = y_0 e^{-0.00012097t}$, With $y(t) = 0.001 y_0$ we have $0.001 y_0 = y_0 e^{-0.00012097t}$ and $-0.00012097t = ln(0.001) = -ln1000$. Thus $t = \frac{ln1000}{0.00012097} \approx 57{,}100$ *years.*

Application 4: The relatively stable uranium-238 is transformed into the isotope plutonium-239 in a breeder reactor. It is found that 0.043% of the initial plutonium amount y_0 had decomposed after 15 years. If the rate of disintegration is proportional to the amount left, get the isotope's half-life.

Solution: Let $y(t)$ denote the amount of plutonium remaining at time t. solution of the initial-value problem $\frac{dy}{dt} = ky, \quad y(0) = y_0$

From properties of HK transform we get $HK\left\{\frac{dy}{dt}\right\} = HK\{ky\}$

And similar to previous applications we can see the solution is $y(t) = y_0 e^{kt}$. If 0.043% of the atoms of y_0 have disintegrated, then 99.957% of the substance remains. To find the decay constant k, we use $0.99957 y_0 = y(15)$ - that is, $0.99957 y_0 = y_0 e^{15k}$. Solving for k then gives $k = \frac{1}{15} ln 0.99957 = -0.00002867$. Hence $y(t) = y_0 e^{-0.00002867t}$. Now the half-life is the corresponding value of time at which $y(t) = \frac{1}{2} y_0$. Solving for t gives $\frac{1}{2} y_0 = y_0 e^{-0.00002867t}$, or $\frac{1}{2} = e^{-0.00002867t}$. The last equation yields so,

$t = \frac{ln2}{0.00002867} \approx 24{,}180 yr$

5. Conclusion

The development of the HK transform in this study has effectively addressed problems of population growth and decay. The applications presented demonstrate the effectiveness of the HK transform in solving these issues. The results of the investigation indicate that the proposed transform produces accurate outcomes without requiring complex computations. In the future, mathematical models for solving a range of challenging problems in science, technology, and medicine could be developed using the HK transform.

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