مجلة جامعة سبها للعلوم البحتة والتطبيقية
Sebha University Journal of Pure & Applied Sciences

Journal homepage: www.sebhau.edu.ly/journal/jopas

Computing Poincaré map for One Parameter Pulsed Dipole System Using Hénon Trick

Ibrahim Alsendid*

Mathematics Department, Faculty of Science, Sebha University, Sebha, Libya.

Keywords:
Hénon trick.
Hénon method.
Poincaré map.
Dipole system.
Lorenz system.

ABSTRACT

For a trajectory generated by dynamical systems, Hénon has presented a method called the Hénon trick or Hénon method. In this method, a surface of a section (Poincaré surface) is defined, and the Poincaré map (i.e., the trajectory points distributed on it) is collected when the trajectory crosses the Poincaré surface. Whenever the Hénon trick is used to calculate the Poincaré map, the autonomous chaotic system's trajectory deviates from the original path, causing a deformation in its attractor. In this paper, the Hénon trick is discussed to calculate the Poincaré map for the attractor Lorenz system, after which a 1-parameter pulsed dipole (source-sink pairs) model is defined on an unbounded domain, and a Python data science code is built to plot the results. The paper provided a reformed Hénon trick to calculate the Poincaré map for a 1-parameter pulsed dipole model by defining a cross-section (Poincaré surface), then I calculate the Poincaré map of the intersection points between this cross-section and the streamlines generated by that pulsed dipole model. The Poincaré map is important to investigate the uniformity of the distribution of streamlines generated by the pulsed dipole system.

حساب تطبيق بو انكاريه لثنائي القطب النبضي ذو معلمة واحدة باستخدام خدعة هينون

إبراهيم الصنديد*

قسم الرياضيات، كلية العلوم، جامعة سبها، سبها، ليبيا.

الكلمات المفتاحية:
طريقة هنون.
خدعة هنون.
تطبيق بوانكاريه.
نظام ثنائي القطب النبضي.
نظام لورينتز.

الملخص

ليكن لدينا جاذب أو مسار (خطوط انسيابية) مولدة بواسطة نظام معادلات ديناميكي، قدم هينون طريقة تعرف بخدعة هينون أو طريقة هينون, في هذه الطريقة يتم فيها تعريف سطح يسمى (سطح بوانكاريه). عندما يعبر مسار الخطوط الانسيابية سطح البوانكاريه ينتج تجمع من النقاط موزعة على المسار تسمى تطبيق بوانكاريه, حيث ينحرف مسار النظام الفوضوي المستقل عن المسار الأصلي مما يؤدي إلى تشوه في المسار أثناء حساب تطبيق بوانكاريه. في هذه الورقة، تم مناقشة خدعة هينون لحساب تطبيق بوانكاريه لنظام جاذب لورينز، وتم تعريف نظام ثنائي القطب النبضي ذو معامل واحد معرف على مجال غير محدود، حيث تم استخدام لغة برمجة البايثون لبناء كود وعرض النتائج. القطب النبضي هو نظام يتكون من قطبين مصدر ومصرف، وهو أول مثال يعرض مسارات الجسيمات الفوضوية مع أنه نظام لتدفق خطوط انسيابية بدون دوران . هذه الورقة تقدم طريقة معدلة لخدعة هينون لحساب تطبيق بوانكاريه لنظام ثنائي القطب النبضي ذو معلمة واحدة معرف على مجال غير محدود. تم تطبيق طريقة هينون من خلال اختيار سطح بوانكاريه، من ثم حساب تطبيق بوانكاريه لنقاط التقاطع بين هذا السطح والخطوط الانسيابية المولدة بواسطة نظام ثنائي القطب النبضي. يعتبر تطبيق البوانكاريه أداة مهمة جداً لدراسة تجانس توزيع الخطوط الانسيابية المولدة بواسطة نظام ثنائي القطب النبضي.

1. Introduction

Michel Hénon in [1] described the Hénon trick which is also called (Hénon method) as follows: For an autonomous dynamical system defined by the following N simultaneous differential equations:

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_N), \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_N), \dots, \frac{dx_N}{dt} = f_N(x_1, x_2, \dots, x_N). \quad (1)$$

A solution can be represented by a curve or trajectory in an N-dimensional phase space $(x_1, x_2, \dots, x_N)$. A frequently used technique

*Corresponding author.
E-mail addresses: Ibr.alsendid@sebhau.edu.ly

Article History : Received 15 March 25 - Received in revised form 17 May 25 - Accepted 21 June 25

consists of considering the successive intersections of the trajectory with a surface of section Σ. Which in general is an $(N-1)$-dimensional subset of the phase space, defined by the following equation:

$$S(x_1, x_2, \dots, x_N) = 0 \quad (2)$$

The dynamical system is defined in equation 1, then a mapping of Σ on itself, known as a Poincaré map. Investigating this mapping is more informative and easier than the examination of the trajectories.

The Poincaré map is defined indirectly by equations 1 and 2. To find an image of a point P of Σ, we follow the trajectory proceeding from P until it intersects Σ again, and the intersections with Σ that is defined by equation 2 is computed. The working mechanism can be explained as follows:

The First step is: to integrate the system in equation 1 with a fixed integration step size, i.e., to obtain a sequence of integration points on the trajectory, and to evaluate S given by equation 2 at each point until a change of sign is detected. The Hénon trick is to arrange the integration scheme in such a way that one integration point lies exactly on Σ. The method considered first the case where equation 2 has the simple form: For a constant number a, $x_1 - a = 0$. (3)

A permutation of coordinates brings this to the form:

$$x_N - a = 0. \quad (4)$$

To obtain an integration point on the form:

$$t - a = 0. \quad (5)$$

Then the surface of the section was defined by a condition on the independent variable t. It is observed that x_N in equation 3 is a dependent variable. Thus, rearrange the differential system by dividing the $(N-1)$ first equations in equation 1 by the last quantity, and inverting the last equation:

$$\frac{dx_1}{dx_N} = \frac{f_1}{f_N}, \dots, \quad \frac{dx_{N-1}}{dx_N} = \frac{f_{N-1}}{f_N}, \quad \frac{dt}{dx_N} = \frac{1}{f_N}. \quad (6)$$

So, t has now become a dependent variable. $\frac{1}{f_N}$ depends on x_N which is an independent variable. In practice, the system in equation 1 is integrated until a change of sign is detected for the quantity $S = x_N - a$. Then the system shifted to equation 6. Using either the last computed point or the previous one as an initial point, the use of the last point produces a slightly simpler program. The system in equation 6 is integrated for one step, taking as an integration step

$$\Delta x_N = -S. \quad (7)$$

Thus, the second step is to stop the integration immediately after the trajectory crosses the Poincaré surface (Σ). x_N is a component of Δx_N which is the distance between the first integration point after crossover and the Poincaré surface. Then, as a third step, we compute the next integration point by integrating equation 6 with a fixed integration step size $-\Delta x_N$.

A current independent variable τ is introduced here to merge the two systems equation 1 and equation 6.

$$K = \frac{dt}{d\tau} \quad (8)$$

Where the general form is given by:

$$\frac{dx_1}{d\tau} = K f_1, \dots, \quad \frac{dx_N}{d\tau} = K f_N, \quad \frac{dt}{d\tau} = K. \quad (9)$$

For $K = 1$, we meet the system equation 1, while equation 6 is obtained when $K = \frac{1}{f_N}$.

Palaniyandi in [2] presents the original attractor Lorenz system of $\sigma = 16$, $\rho = 45.92$, $\beta = 4$. Then he provided an example for computing the Poincaré map for the system using Hénon trick.

In this research, a model of the pulsed dipole system will be studied. The pulsed dipole system was the first illustration of a flow without circulation with chaotic particle pathways. In an unbounded plane, Jones and Aref in [3] provided a system that consisted of a simple potential flow model, which has been used to describe the flow from the source to the sink. e.g., a single source-sink pair operating when the source is on, the sink will be on, with fluid injected by the sink being ousted by the source. It is also highlighted that the fluid is extracted at the sink and then reinjected at the source on the same streamline as it entered the sink.

One way to investigate the behaviour of the streamlines generated by a pulsed dipole system is to use the Hénon trick, which requires an accurate computational methodology and demonstrate the results using Python data science.

The paper is organised as follows; the Lorenz system will be introduced in section 2. Then, section 3 will discuss a modified Hénon algorithm that is provided in [2] to construct the attractor of the Lorenz equations and calculate the Poincaré map.

In section 4, a reformed Hénon trick will be provided to calculate the Poincaré map for the original attractor Lorenz system, essentially by computing the forward orbit of an initial condition, where the plotted Poincaré map will be compared with Palaniyandi's method.

In section 5, a 1-parameter pulsed dipole (source-sink pairs) model $Dipole(a, 0)$ will be defined on the unbounded domain, which is an advection generated by the pulsed dipole (source–sink pairs) model.

Lastly, in section 6, the reformed Hénon trick will be utilised to calculate the Poincaré map for a 1-parameter pulsed dipole model, i.e., the intersection of the streamlines generated by $Dipole(a, 0)$ with a cross-section curve. That is a tool to investigate the uniformity of the distribution of streamlines on the manifold.

2. The Lorenz System

Edward Norton Lorenz (1917–2008) provided a system of differential equations in 1963 to explain some of the weather's behaviors. Even though most possible models for predicting the weather require PDE, Lorenz provided a simpler system as provided in equations (10 – 12).

$$\frac{dx}{dt} = \sigma(y - x), \quad (10)$$

$$\frac{dy}{dt} = x(\rho - z) - y, \quad (11)$$

$$\frac{dz}{dt} = xy - \beta z, \quad (12)$$

This model is the Lorenz equations, which is a system of three ordinary differential equations, where y corresponds to the horizontal temperature variation, x to the rate of convection, and z corresponds to the vertical temperature variation. The system parameters $\rho, \beta,$ and σ correspond to the Rayleigh number, physical dimensions of the layer, and Prandtl number.

The non-linearity of the equations of flow causes non-linearity in the two equations (11) and (12) [4]. By a resourceful argument, Lorenz concluded that the Lorenz attractor looks like a single surface.

Lorenz butterfly attractor for the parameters $\sigma = 10$, $b = 28$, and $r = 8/3$ is investigated by [4 - 9]. While a view of the Rössler band attractor for the parameters $a = 0.173$, $b = 0.4$, c changes from 0 to 7 is studied by [7] and [9]. Moreover, Yan et al in [8] expanded the conventional Lorenz system to include fractal and fractional dynamics, and provides a numerical analysis of its chaotic behaviour.

3. Modified Hénon Algorithm Provided by (Palaniyandi 2009)

Palaniyandi in his paper [2] provided a modified Hénon algorithm to calculate the Poincaré map of the original attractor Lorenz system where $\sigma = 16$, $\rho = 45.92$, $\beta = 4$ crossed the Poincaré surface ($z - 44.92 = 0$). Palaniyandi's algorithm of the modified Hénon method is as follows:

- Integrated equation 1 utilizing a step size of $h = 0.005$.
- The integration is stopped once the trajectory reaches the Poincaré surface (Σ). The function Δx_N computes the distance x_N between the first integration point after crossover and the Poincaré surface.
- Then the values of all variables x_i where ($i = 1, 2, \dots, N$) are stored.
- The next integration point is obtained by integrating equation (6) with a step size of $-\Delta x_N$.
- All variables were reset to the values recorded in the third step.
- Lastly, the system (1) kept integrating with step size $h = 0.005$.

Hence, the Poincare´ map is plotted as shown in figure 1.

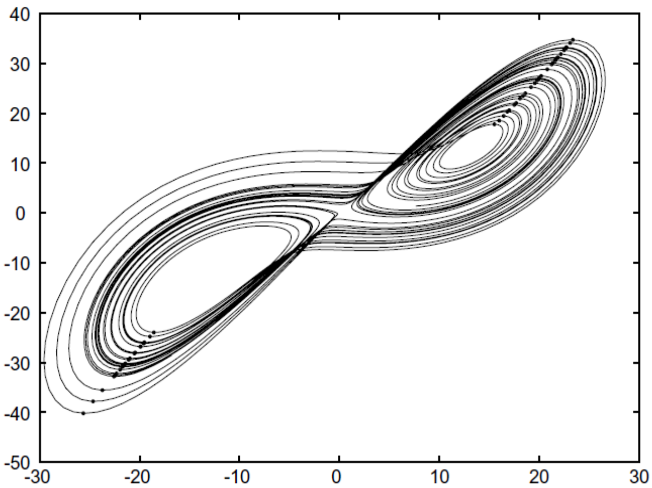

Fig. 1: The Poincaré map for the Lorenz system using the modified Hénon algorithm cited in [2].

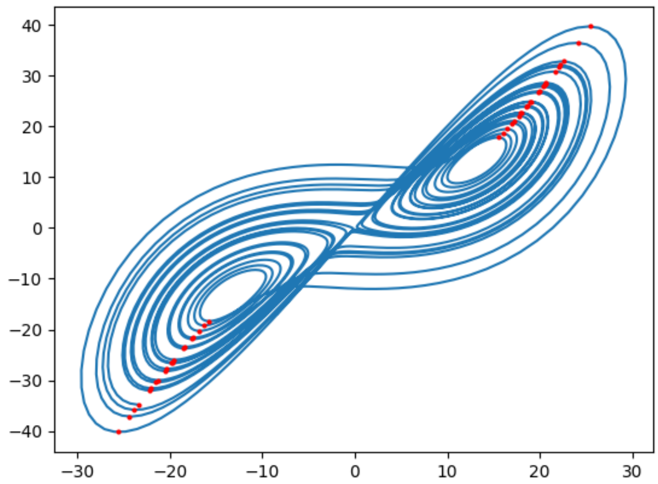

Fig. 2: Shows the Poincaré map for Lorenz system calculated by the Reformed Hénon Trick.

4. Reformed Hénon Trick

In this paper, an algorithm of the reformed Hénon trick is provided as follows, where equation 13 presents the original attractor Lorenz system of $\sigma = 16, \rho = 45.92, \beta = 4$.

$$\frac{dx}{dt} = \sigma(y - x), \frac{dy}{dt} = \rho x - y - xz, \frac{dz}{dt} = xy - \beta z, \qquad (13)$$

Then, by dividing the first and second components of the equation 13 by the third component, and inverting the third component we get equation 14.

$$\frac{dx}{dz} = \frac{\sigma(y - x)}{xy - \beta z}, \frac{dy}{dz} = \frac{\rho x - y - (xz)}{xy - \beta z}, \frac{dt}{dz} = \frac{1}{xy - \beta z}. \qquad (14)$$

- Integrate equation 13 to move forward with a fixed integration step size $h = 0.005$ (solve Lorenz system using Runge-Kutta integrator).
- Stop the integration immediately after the trajectory crosses the Poincaré surface $(z - 44.92 = 0)$, and restore the cross point p'.
- Compute the next integration point by integrating equation 14 starting from the cross point p' and moving backward until we meet the Poincaré surface $(z = 44.92)$, store the integration point, that is the intersection point P.
- Then equation 13 is integrated from the intersection point P with a step size $h = 0.005$.
- Kept integrating till we get another cross point $p_2{}'$.
- The points $P_1, P_2, P_3, \ldots$ are collectively called as Poincaré map (these points are also called Poincaré points).

The integration points computed from intersecting the Lorenz trajectory with a Poincaré surface of section Σ that is defined by $z - 44.92 = 0$. To collect the integration points P_i, where i denotes the point obtained during ith surface crossing, and $i = 1, 2, 3, \ldots$. The trajectory of the system equations (10 – 12) is integrated until it crosses and intersects the Poincaré surface (Σ). Then Δz is the distance alongside z direction between the first integration point P'_i of the system in equations (10 – 12) after its trajectory crosses over this surface and the Poincaré surface (Σ), then the integration is stopped.

If $\Delta z = 0$, then it means that the point of intersection of the trajectory and the Poincaré surface is present on the trajectory itself, it should be noted down as in figure 2.

The Poincaré map for Lorenz system that calculated by my reformed Hénon trick is validated by comparing the result of this paper in figure 2 with Palaniyandi's result in figure 1.

5. Define an Unbounded 1-Parameter Pulsed Dipole System

The pulsed dipole system is activated to produce mixing as an example of a perturbed map in which chaos is produced in a dynamical system. The most significant thing is that the two source-sink pairs system can be modelled with a pair of non-monotonic shears map.

Here, the unbounded pulsed dipole system will be defined by explaining the working mechanism and the potential flow of the system, then the Poincaré map for the 1-parameter pulsed dipole model $Dipole(a, 0)$ will be computed.

5.1. Potential Flow of Pulsed Dipole System

In the case of the unbounded domain, potential flow (imaginary and real parts of complex potential) can be presented by the following equation quoted by [10] and [11].

$$F(z) = \phi + i\psi = \frac{g}{2\pi}\left(log(z - z_{\oplus}) - log(z - z_{\ominus})\right) \qquad (15)$$

For a period of time t, the movement of the streamlines in the plane is ruled by equation 15, which will introduce a chaotic advection to the system by periodically switching operation into the source-sink pair systems. $(z_{\oplus 1}, z_{\ominus 1})$ and $(z_{\oplus 2}, z_{\ominus 2})$ are two source-to-sink pairs. The flow moves forward under the first source-sink pair from ($z_{\oplus 1}$ towards $z_{\ominus 1}$) for a pump time (step-size) and stops (a half-cycles), then governed by the second source-sink pair from ($z_{\oplus 2}$ towards $z_{\ominus 2}$) for one pump time (another half-cycles) to complete one iteration time, it is advected for iteration time t and switching periodically between the two pairs. The flow comes out from the domain, such as $z_{\oplus 1}$, through a sink such as $z_{\ominus 1}$ and is re-injected through a source during the next cycle of operation.

Following equation 15, we can present the horizontal and vertical source-sink pairs (dipole) system. The horizontal source-sink pair, that is when $z_{\oplus 1} = (-a, 0), z_{\ominus 1} = (a, 0)$, that is for a source $S_{\oplus}$ of strength g, placed at $(-a, 0)$ in the complex plan, and the corresponding sink $S_{\ominus}$ of strength $-g$ at $(a, 0)$. For time t, $z = x(t) + iy(t)$, the velocity potential ϕ is the real part of the complex potential, while the stream function ψ is the imaginary part. The equations of motion are then given by: $(\dot{x}(t), \dot{y}(t)) = \nabla\phi = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$.

$$(\dot{x}, \dot{y}) = \left(\frac{x + a}{(x + a)^2 + y^2} - \frac{x - a}{(x - a)^2 + y^2}, \frac{y}{(x + a)^2 + y^2} - \frac{y}{(x - a)^2 + y^2}\right) \qquad (16)$$

5.2. The 1-Parameter Pulsed Dipole Model $Dipole(a, 0)$

Here, a 1-parameter pulsed dipole (source-sink pairs) model $Dipole(a, 0)$ is represented on an unbounded domain, where $a = 0.5$.

For the horizontal source-sink pair the source is placed at $(-0.5, 0)$ and the sink is placed at $(0.5, 0)$, while the vertical source-sink pair is from $(0, -i\ 0.5)$ to $(0, i\ 0.5)$, with a pump time 0.556 and iteration time 10000 as shown in figure 3.

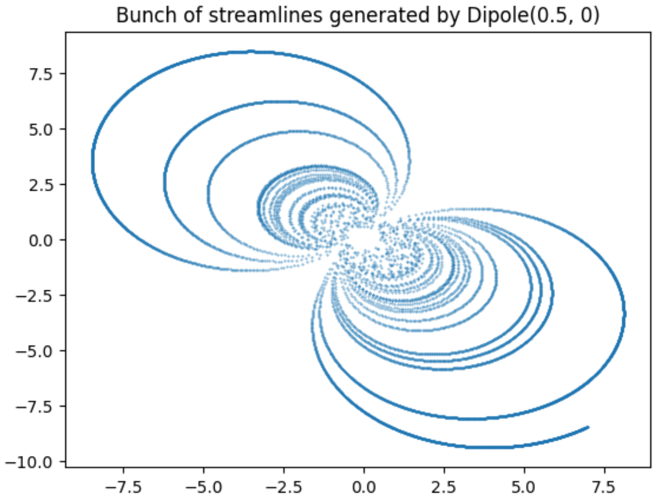

Fig. 3: Shows a bunch of streamlines generated by $\boldsymbol{Dipole(a, 0)}$ where $\boldsymbol{a = 0.5}$ at iteration time 10000.

6. Computing Poincaré Map for Dipole Model $\boldsymbol{Dipole(a, 0)}$

For an unbounded domain pulsed dipole model defined and parameterized above where the system $Dipole(a, 0)$ built with two source-sink pairs in one parameter $a = 0.5$, the first source-sink pair is from $z_{\oplus 1} = (-0.5, 0)$ towards $z_{\ominus 1} = (0.5, 0)$, and the second source-sink pair is from $z_{\oplus 2} = (0, -i\ 0.5)$ toward $z_{\ominus 2} = (0, i\ 0.5)$.

For a cross-section curve $\gamma = \{(x, y) : x = 0\}$ that is a circumference, a normalized intersection measure describes the local distribution of intersection points of γ with the streamlines generated by the 1-parameter pulsed dipole model $Dipole(0.5, 0)$ on an unbounded domain using the Hénon trick.

The Poincaré map is defined implicitly by intersecting $Dipole(0.5, 0)$ with the cross-section γ, and plotted at iteration times 4000 and 10000 as shown in figure 4 and figure 5 respectively.

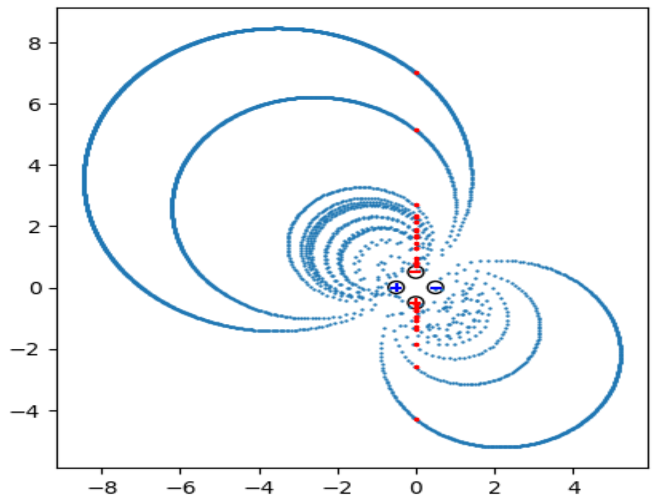

Fig. 4: Shows the Poincaré map for the pulsed dipole model $\boldsymbol{Dipole(0.5, 0)}$ at iteration time 4000, calculated by the Reformed Hénon Trick.

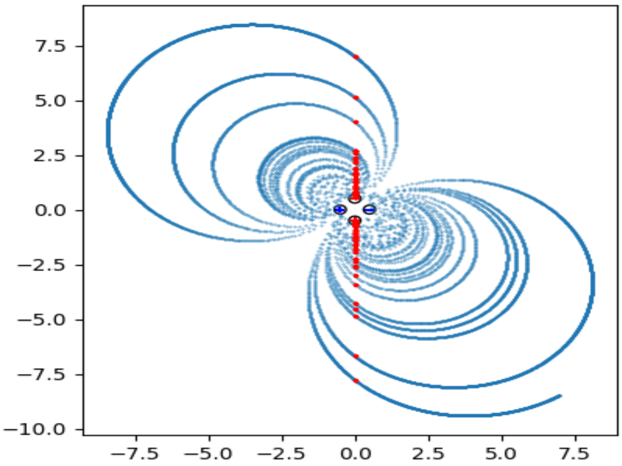

Fig. 5: Shows the Poincaré map for the pulsed dipole model $\boldsymbol{Dipole(0.5, 0)}$ at iteration time 10000, calculated by the Reformed Hénon Trick.

7. Conclusion

The paper provided a reformed Hénon trick and utilized it to calculate the Poincaré map for the original attractor Lorenz system, where the plotted Poincaré map was compared with Palaniyandi's method in section 4. Moreover, the paper defined a 1-parameter pulsed dipole model $Dipole(a, 0)$ on an unbounded domain. Then, the reformed Hénon trick was explained and employed to calculate the Poincaré map for $Dipole(a, 0)$, where the streamlines generated by $Dipole(a, 0)$ intersect the Poincaré surface $(x = 0)$. The Poincaré map is plotted and visualized on the streamlines.

Further study could investigate the behaviour of the pulsed dipole system using the results of this paper.

8. Python Code

The GitHub link for the Python code:
https://github.com/IbrahimAlsendid/Reformed-H-non-trick-to-calculate-the-Poincar-map.git

9. References

[1] Hénon, M. (1982). On the numerical computation of Poincaré maps. Physica D: Nonlinear Phenomena, 5(2-3), 412-414.

[2] Palaniyandi, P. (2009). On computing Poincaré map by Hénon method. Chaos, Solitons & Fractals, 39(4), 1877-1882.

[3] Jones, S. W., & Aref, H. (1988). Chaotic advection in pulsed source–sink systems. The Physics of fluids, 31(3), 469-485.

[4] Domínguez-Tenreiro, R., Roy, L. J., & Martinez, V. J. (1992). On the multifractal character of the Lorenz attractor. Progress of theoretical physics, 87(5), 1107-1118.

[5] Pinsky, T., (2024). New Knots in the Lorenz Equations. Notices of the American Mathematical Society, 71(10).

[6] Macek, Wies law M (2012). "Fractals and Multifractals". In: Courses at Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszynski University, Warsaw, Poland.

[7] Hirsch, Morris W, Smale, Stephen and Devaney, Robert L (2012). Differential equations, dynamical systems, and an introduction to chaos. Academic press.

[8] Yan, T., Alhazmi, M., Youssif, M.Y., Elhag, A.E., Aljohani, A.F. and Saber, S., 2024. Analysis of a Lorenz model using adomian decomposition and fractal-fractional operators. Thermal Science, 28(6 Part B), pp.5001-5009.

[9] Falconer, K. (2014). Fractal geometry: mathematical foundations and applications. John Wiley & Sons.

[10] Hertzsch, J. M., Sturman, R., & Wiggins, S. (2007). DNA microarrays: design principles for maximizing ergodic, chaotic mixing. Small, 3(2), 202-218.

[11] Stremler, M. A., & Cola, B. A. (2004). Chaotic Advection and Mixing in Pulsed Source-Sink Systems. XXI ICTAM, Warsaw, Poland, Aug. 15, 21.